M.Sc III Sem.
Paper - XI.

Quotient space :— The linear space L/M as is called the Quotient space of L Modulo M. The mapping ϕ of L onto L/M defined by $\phi(x) = x+M$ is called the canonical mapping of L onto L/M.

Theorem :— Let M be a closed linear manifold (subspace) in a normed linear space N. For each coset $x+M$ in the quotient space N/M we define

$$\|x+M\| = \inf\{\|x+M\| : m \in M\}.$$

Then $\|x+M\|$ is a norm on N/M and thus N/M is a normed linear space. Further if N is a Banach space, then so in N/M.

Proof:— We verify all the postulates for a norm.

(n1): Since $\|x+m\|$ is a non-negative real number and every set of non-negative real numbers is bounded below, it follows that $\inf\{\|x+m\| : m \in M\}$ exists and is non-negative, that is

$$\|x+M\| \geq 0 \ \forall \ x+M \in N/M$$

(n2): Let $x+M = M$ (the zero element of N/M). Then $x \in M$.

Hence $\|x+M\| = \inf\{\|x+M\| : m \in M, x \in M\}$.
$= \inf\{\|y\| : y \in M\} = 0$

[$\because$ M being a subspace contains zero vector whose norm is real number 0]

~~Then~~ Thus $x+M = M \Rightarrow \|x+M\| = 0$

Conversely, we have

$\|x+M\| = 0 \Rightarrow \inf\{\|x+m\| : m \in M\} = 0$

$\Rightarrow$ there exists a sequence $\langle m_k \rangle_{k=1}^{\infty}$ in M

Such that $\|x+m_k\| \to 0$ as $k \to \infty$.

$\Rightarrow \lim_{k\to\infty} m_k = -x$

$\Rightarrow -x \in M$ (since M is closed and $\langle m_k \rangle$ is a sequence in M converging to $-x$) —— (1)

$\Rightarrow x \in M$ ($\because M$ is subspace)

$\Rightarrow x+M = M$ (the zero element of N/M)

thus we have show that

$\|x+M\| = 0 \Rightarrow x+M = M$ (the zero of N/M)

(n3): let $x+M$, $y+M \in N/M$. Then

$\|(x+M)+(y+M)\| = \|(x+y)+M\|$

by definition of addition of coset.

$= \inf\{\|x+y+m\| : m \in M\}$ —— (2)

$= \inf\{\|x+y+m+m'\| : m \in M, m' \in M\}$ —— (3)

[$\because$ M is subspace, the sets in (2) and (3) are the same]

$= \inf\{\|(x+m+(y+m')\| : m, m' \in M\}$

$\leq \inf\{\|x+m\| + \|y+m'\| : m, m' \in M\}$.

Using (n3) for N since $x+m, y+m' \in N$

$= \inf\{\|x+m\| : m \in M\} + \inf\{y+m'\| : m \in M\}$.

$= \|x+M\| + \|y+M\|$

(n4): $\|\alpha(x+M)\| = \inf\{\alpha\|x+m\| : m \in M\}$

$= \inf\{|\alpha|\,\|x+m\| : m \in M\}$

by (n4) for N.

$= |\alpha| \inf \{ \|x + m\| : m \in M \}$

$= |\alpha| \ \|x + M\|$

Hence N/M is a normed linear space.

Now, we prove that if N is complete, then so is N/M.

We know that a Cauchy sequence is convergent if and only if it has a convergent subsequence.

Let $\langle S_n + M \rangle_{n \geq 1}^{\infty}$ be Cauchy sequence in N/M

So that $S_n \in N$.

We construct a convergent subsequence of this sequence as follows:

Since $\langle S_n + M \rangle$ is a Cauchy sequence, for $\epsilon = 1/2$ there exists a positive integer n_1 such that

$n, m \geq n_1 \Rightarrow \|(S_n + M) - (S_m + M)\| < \frac{1}{2}$

Set $S_{n_1} = x_1 \in N$.

Similarly for $\epsilon = (\frac{1}{2})^2$ there exists a positive integer $n_2 > n_1$ such that

$n, m \geq n_2 \Rightarrow \|(S_n + M) - (S_m + M)\| < (\frac{1}{2})^2$

Set $S_{n_2} = x_2 \in N$.

In general having chosen $x_1, x_2, \ldots, x_k$ and $n_1, n_2, \ldots, n_k$. Let $n_{k+1} > n_k$ be such that $n, m \geq n_k \Rightarrow \|(S_n + M) - (S_m + M)\| < (\frac{1}{2})^{k+1}$, set $x_{k+1} = S_{n_{k+1}}$.

Thus we have obtained a subsequence $\langle x_k + M \rangle_{k=1}^{\infty}$ of $\langle S_n + M \rangle$

such that $\|(x_{k+1} + M) - (x_k + M)\| < (\frac{1}{2})^k$.

$k = 1, 2, 3, \ldots$

It will now be shown that this subsequence

Converges to an element of N/M. We begin by choosing any vector y_1 in x_1+M.

After choosing y_1, we select $y_2 \in x_2+M$ such that $\|y_1 - y_2\| < \frac{1}{2}$.

This is possible for $y_1 \in x_1+M \Rightarrow y_1 = x_1+m_1$ for some $m_1 \in M$ and $\|(x_1+M)-(x_2+M)\| < \frac{1}{2} \Rightarrow$

$\inf\{\|x_1-x_2+m\| : m \in M\} < \frac{1}{2} \Rightarrow \exists$ an $m_0 \in M$ such that

$\|x_1-x_2+m_0\| < \frac{1}{2}$

$\Rightarrow \|y_1-y_2\| < \frac{1}{2}$ where $y_2 = x_2+m_0+m_1 \in x_2+M)$. We next select y_3 in x_3+M such that $\|y_2-y_1\| < (\frac{1}{2})^2$.

Continuing in this manner we get a sequence $\langle y_n \rangle$ in N such that

$$\|y_n - y_{n+1}\| < \left(\tfrac{1}{2}\right)^n \quad \text{——— (4)}$$

We claim that, $\langle y_n \rangle$ is a Cauchy sequence in N. For if $\epsilon > 0$ be given, we can choose a positive integer m_0 so large that

$$\frac{1}{2^{m_0-1}} < \epsilon$$

Then for $n > m \geq m_0$, we have

$$\|y_m - y_n\| = \|(y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \text{—} +$$
$$(y_{n-1} - y_n)\| \leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \text{—} + \|y_{n-1} - y_n\|$$

$$\leq \sum_{i=m}^{n-1} (1)^i \quad \text{by (4)}$$

$$\sum_{i=m}^{\infty} \left(\tfrac{1}{2}\right)^i = \left(\tfrac{1}{2}\right)^m + \left(\tfrac{1}{2}\right)^{m+1} + \text{———}$$

$$= \frac{(1/2)^m}{1-\frac{1}{2}} = \frac{1}{2^{m-1}} \leq \frac{1}{2^{m_0-1}} < \epsilon$$

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Hence $\langle y_n \rangle$ is a Cauchy sequence in N and Since N is Complete, there exists a vector y in N such that $y_n \to y$.

Finally we prove that

$$x_n + M \to y + M \in N/M. \text{ We have}$$

$$\|(x_n+M) - (y+M)\| = \|x_n - y + M\|$$

$$= \inf\{\|x_n - y + m\| : m \in M\}$$

$$\leq \|x_n + m - y\| \text{ for all } m \in M.$$

Since $y_n = x_n + m_n$ for some $m_n \in M$, we conclude that

$$\|(x_n+M) - (y+M)\| \leq \|y_n - y\| \to 0 \text{ since } y_n \to y$$

Hence $x_n + M \to y + M \in N/M$ and consequently N/M is Complete.

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